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EQUATIONS OF THE NEHER-HARPER CIRCUIT

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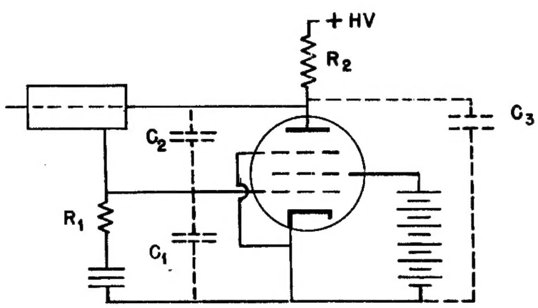
INTRODUCTION

An equation expressing the plate potential in a Neher-Harper circuit as a function of time has been discussed in Report CP-3609. Report CP-3609 also contains data verifying the equation. Let us now see how the equation may be derived.

STATEMENT OF THE PROBLEM



The following assumptions are made:

1. The circuit is linear. (A vacuum tube operated near cutoff is not very linear. After we have derived a solution we shall discuss the effects of nonlinearity.)

2. Grid current is neglected. (This assumption also open to question.)

3. The leakage resistance across the G-M tube is infinite.

4. A charge q in coulombs is deposited on the grid at t = o, and a charge -q is at the same time deposited on the plate.

The diagram shows the meaning of the symbols. It follows from Kirchoff's current law that

$$(C_1 + C_2)\frac{dv_1}{dt} + \frac{v_1}{R_1} - C_2\frac{dv_2}{dt} = 0$$

$$C_2\frac{dv_1}{dt} - gm\, v_1 - (C_2 + C_3)\frac{dv_2}{dt} - \left(\frac{1}{R_3} + \frac{1}{r_p}\right) v_2 = 0$$

For convenience let us write

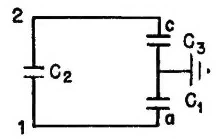
$$A = C_1 + C_2$$

$$B = C_2 + C_3$$

$$\frac{1}{R_4} = \frac{1}{R_3} + \frac{1}{r_p}$$

The next step is to determine the initial conditions.

Let us consider the capacitances of the circuit apart from the resistances.



The total capacitance between points 1 and 2 is

$$C_2 + \frac{C_1 \quad C_3}{C_1 + C_3} = \frac{C_1 C_2 + C_2 C_3 + C_1 C_3}{C_1 + C_3}$$

Therefore the potential difference $[v_1 (0) - v_2 (0)]$ between 1 and 2 is

$$q \frac{C_1 + C_3}{C_1 C_2 + C_2 C_3 + C_1 C_3}$$

Now the charge on condenser plate a equals the charge on condenser plate c. Therefore

$$\frac{v_1 (0)}{- v_2 (0)} = \frac{C_3}{C_1}$$

From this we see that

$$v_1(0) = q \left(\frac{C_3}{C_1 C_2 + C_2 C_3 + C_1 C_3}\right)$$

$$- v_2(0) = - q \left(\frac{C_1}{C_1 C_2 + C_2 C_3 + C_1 C_3}\right)$$

GENERAL SOLUTION OF DIFFERENTIAL EQUATIONS

The differential equations are

$$\left\{\begin{array}{l} A \dfrac{dv_1}{dt} + \dfrac{v_1}{R_1} - C_2 \dfrac{dv_2}{dt} = 0 \\[2ex] C_2 \dfrac{dv_1}{dt} - g_m \; v_1 - B \dfrac{dv_2}{dt} - \dfrac{v_2}{R_4} = 0 \end{array}\right\}$$

Determinants may be used to eliminate one variable. If the differential operator D denotes one differentiation with respect to time, then

$$\begin{vmatrix} AD + \frac{1}{R_1} & - C_2 D \\ C_2 D - g_m & - BD - \frac{1}{R_4} \end{vmatrix} v_2(t) = 0$$

A similar argument would show that v_1 must satisfy the same differential equation as v_2.

The general solution of this equation is

$$v_2(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

In this solution

$$s_1 = \frac{1}{2}\left[- M + \sqrt{M^2 - 4N} \right]$$

$$s_2 = \frac{1}{2}\left[- M - \sqrt{M^2 - 4N} \right]$$

$$M = \frac{R_1 A + R_4 B + g_m R_1 R_4 C_2}{R_1 R_4 (AB - C_2^2)}$$

$$N = \frac{1}{R_1 R_4 (AB - C_2)}$$

K_1 and K_2 are constants of integration. Furthermore,

$$v_1(t) = K_3 e^{s_1 t} + K_4 e^{s_2 t}$$

in which K_3 and K_4 are constants of integration.

PARTICULAR SOLUTION

Since

$$v_2(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$\frac{dv_2}{dt} = K_1 s_1 e^{s_1 t} + K_2 s_2 e^{s_2 t}$$

Since the differential equations must be satisfied as t approaches zero from the right

$$A \frac{dv_1}{dt}(0) + \frac{v_1(0)}{R_1} - C_2 \frac{dv_2}{dt}(0) - \frac{v_2(0)}{R_4} = 0$$

$$C_2 \frac{dv_1}{dt}(0) - g_m v_1(0) - B \frac{dv_2}{dt}(0) - \frac{v_2(0)}{R_4} = 0$$

$$\frac{dv_2}{dt}(0) = \frac{\begin{vmatrix} A & \dfrac{-v_1(0)}{R_1} \\ C_2 & g_m\, v_1(0) + \dfrac{v_2(0)}{R_4} \end{vmatrix}}{\begin{vmatrix} A & -C_2 \\ C_2 & -B \end{vmatrix}}$$

$$= \frac{Ag_m\, v_1(0) + A\,\dfrac{v_2(0)}{R_4} + \dfrac{C_2}{R_1}\, v_1\,(0)}{-A\,B + C_2^2}$$

$$= \frac{A\,R_1\,R_4\,g_m\,\,v_1(0) + A\,R_1\,v_2\,(0) + R_4\,C_2\,v_1\,(0)}{R_1\,R_4\,(-A\,B + C_2^2)}$$

$$= \frac{-A\,R_1\,q\,C_1 - A\,R_1\,R_4\,g_m\,q\,C_3 - R_4\,C_2\,q\,C_3}{R_1\,R_4\,(AB - C_2^2)^2}$$

$$= -\,q\,\frac{-A\,R_1\,C_1 + (A\,R_1\,R_4\,g_m + R_4\,C_2)\,C_3}{R_1\,R_4\,(A\,B - C_2^2)^2}$$

$$= -\,q\,\frac{-A\,R_1\,C_1 + (A\,R_1\,R_4\,g_m + R_4\,C_2)\,C_3}{R_1\,R_4\,(A\,B - C_2^2)^2}$$

$$\frac{q\,C_1}{A\,B - C_2^2} = K_1 + K_2$$

$$-\,q\,\frac{-A\,R_1\,C_1 + (A\,R_1\,R_4\,g_m + R_4\,C_2)\,C_3}{R_1\,R_4(\,AB - C_2^2)^2} = K_1\,s_1 + K_2\,s_2$$

Or, writing P and Q for the multipliers of q,

$$K_1 + K_2 = -\,q\,P$$

$$s_1\,K_1 + s_2\,K_2 = -\,q\,Q$$

$$K_1 = \frac{\begin{vmatrix} -\,q\,P & 1 \\ -\,q\,Q & s_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ s_1 & s_2 \end{vmatrix}} = q\,\frac{P\,s_2 - Q}{s_1 - s_2}$$

$$K_2 = \frac{-\,q\begin{vmatrix} 1 & P \\ s_1 & Q \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ s_1 & s_2 \end{vmatrix}} = \frac{q(Q - P\,s_1)}{s_1 - s_2}$$

Therefore

$$v_2(t) = -\frac{q}{s_1 - s_2}\left[(-P s_2 + Q)\, e^{s_1 t} + (P s_1 - Q)\, e^{s_2 t}\right]$$

Also

$$\frac{d v_1}{d t}(0) = \frac{\begin{vmatrix} v_1(0) & -C_2 \\ g_m v_1(0) + \dfrac{v_2(0)}{R_4} & -B \end{vmatrix}}{\begin{vmatrix} A & -C_2 \\ C_2 & -B \end{vmatrix}}$$

$$= \frac{-\dfrac{B}{R_1} v_1(0) - g_m C_2 v_1(0) - \dfrac{C_2}{R_4} v_2(0)}{A B - C_2^2}$$

$$= \frac{-R_4 B v_1(0) - R_1 R_4 g_m C_2 v_1(0) - R_1 C_2 v_2(0)}{R_1 R_4 (AB - C_2^2)}$$

$$= q\, \frac{-R_4 B C_3 - R_1 R_4 g_m C_2 C_3 + R_1 C_2 C_1}{R_1 R_4 (AB - C_2^2)^2}$$

Let us write

$$R = \frac{C_3}{A B - C_2^2}$$

$$S = \frac{-R_4 B C_3 - R_1 R_4 g_m C_2 C_3 + R_1 C_2 C_1}{R_1 R_4 (A B - C_2^2)^2}$$

Then

$$K_3 + K_4 = + qR$$

$$s_1 K_3 + s_2 K_4 = + qS$$

$$K_S = \frac{q \begin{vmatrix} R & 1 \\ S & s_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ s_1 & s_2 \end{vmatrix}}$$

$$K_3 = q\, \frac{R s_2 - S}{s_2 - s_1}$$

$$K_4 = \frac{q \begin{vmatrix} 1 & R \\ s_1 & S \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ s_1 & s_2 \end{vmatrix}}$$

$$K_4 = q\,\frac{S - R\,s_1}{s_2 - s_1}$$

$$\therefore\ v_1\,(t) = \frac{q}{s_2 - s_1}\,\left[\,(R\,s_2 - S)\,e^{s_1 t} + (S - R\,s_1)e^{s_2 t}\right]$$

DISCUSSION OF RESULTS

The equation just derived for $v_1(t)$ and $v_2(t)$ may be simplified. First, in most cases M^2 is much larger than 4N. For instance, if R_1 = 5 megohms, $C_1 = 15\mu\mu$ f, $C_2 - 3\mu\mu$ f, R_4 = 1 megohm, and $C_3 = 15\mu\mu$ f, and $g_m = 10^{-4}$ mhos, then

$$M^2 = 6.10^{11}/\text{sec}^2$$

$$4N = 2.10^9/\text{sec}^2$$

Therefore it is approximately true, if $M^2 - 4N$ is expanded, that

$$s_1 = \frac{1}{2}\,(-M - M - 2N) = -\,\frac{N}{M}$$

$$s_2 = -\,M$$

Furthermore, the term $g_m R_1 R_4 C_2$ in the numerator of M is much larger than the other terms. Therefore, we may also say

$$s_1 = -\,\frac{1}{g_m R_4 R_1 C_2}$$

$$s_2 = -\,\frac{g_m C_2}{(AB - C_2^2)}$$

Since s_2 is much larger than s_1, we may say that

$$s_1 - s_2 = \frac{g_m C_2}{AB}$$

Again,

$$P = \frac{C_1}{A\,B - C_2^2} \approx \frac{C_1}{A\,B}$$

$$Q = \frac{-\,A\,R_1\,C_1 + (A\,R_1\,R_4\,g_m + R_4\,C_2)\,C_3}{R_1\,R_4\,(A\,B - C_2^2)^2} \approx \frac{g_m\,C_3}{A\,B^2}$$

$$P\,s_2 \approx -\,\frac{g_m\,C_1\,C_2}{A^2\,B^2}$$

$$P\,s_1 \approx -\,\frac{C_1}{g_m\,R_1\,R_4\,C_2\,A\,B}$$

Then, using these approximate expressions, we see that

$$v_2\,(t) = \frac{q\;A\,B}{g_m\,C_2}\left[-\left(\frac{g_m C_3}{A\,B^2} + \frac{g_m\,C_1\,C_2}{A^2\,B^2}\right)e^{s_1 t} + \left(\frac{C_1}{g_m\,R_1\,R_4\,C_2\,A\,B} + \frac{g_m C_3}{A\,B^2}\right)e^{s_2 t}\right]$$

$$= q\left[-\left(\frac{C_3}{C_2\,B} + \frac{C_1}{A\,B}\right)e^{s_1 t} + \left(\frac{C_1}{g_m R_1 R_4 C_2^2} + \frac{C_3}{C_2\,B}\right)e^{s_2 t}\right]$$

then, if $C_1 \gg C_2$ and $C_3 \gg C_2$,

$$v_2(t) = \frac{q}{C_2}\left[-e^{s_1 t} + e^{s_2 t}\right]$$

A similar argument leads to the following approximate equation:

$$v_1(t) = q\left[\frac{1}{g_m R_4 C_2}\; e^{s_1 t} + \frac{C_3}{A\,B}\; e^{s_2 t}\right]$$

Figure 1 shows $v_1(t)$ and $v_2(t)$ as calculated for a particular case by means of the equations just derived. Nonlinearity of the vacuum tube means that g_m is not a constant. A reasonable procedure is to use in our equations the mean value which g_m has during a pulse. The maximum value of $v_2(t)$ will be rather accurately proportional to q. The exponents s_1 and s_2 will vary with q. The grid bias of the tube also affects s_1 and s_2.

The effects of grid current may be represented by an effective resistance in shunt with the grid resistor R_1, if the changes in the grid potential v_1 are small. It is clear, however, that the grid may go positive with respect to the cathode for an interval of a microsecond or less at the beginning of the pulse. The grid current during this interval is a problem in itself.

SUMMARY

Symbols:

R_1 = grid resistance in ohms, $A = C_1 + C_2$

C_1 = grid-ground capacitance in farads, $B = C_2 + C_3$

C_2 = grid-plate capacitance in farads

C_3 = plate-ground capacitance in farads

R_3 = load resistance in ohms

r_p = plate resistance of tube in ohms

g_m = transconductance of tube in mhos

q = charge in coulombs passing through the counter in one pulse

$v_1(t)$ = grid potential in volts at the time t, t being in seconds

$v_2(t)$ = plate potential in volts at the time t

Exact solution:

$$v_1(t) = \frac{q}{s_2 - s_1}\left[(R\,s_2 - S)e^{s_1 t} + (S - R\,s_1)\,e^{s_2 t}\right]$$

$$v_2(t) = \frac{q}{s_2 - s_1}\left[(Q - P\,s_2)\,e^{s_1 t} + (P\,s_1 - Q)\,e^{s_2 t}\right]$$

in which

$$R = \frac{C_3}{A\,B - C_2^2}$$

$$S = \frac{-R_4 B C_3 - R_1 R_4 g_m C_2 C_3 + R_1 C_1 C_2}{R_1 R_4 (A B - C_2^2)^2}$$

$$P = \frac{C_1}{A B - C_2^2}$$

$$Q = \frac{-A R_1 C_1 + A R_1 R_4 g_m C_3 + R_4 C_2 C_3}{R_1 R_4 (A B - C_2^2)^2}$$

$$s_1 = \frac{1}{2}\left(-M + \sqrt{M^2 - 4N}\right)$$

$$s_2 = \frac{1}{2}\left(-M - \sqrt{M^2 - 4N}\right)$$

$$M = \frac{A R_1 + B R_4 + g_m R_1 R_4 C_2}{R_1 R_4 (A B - C_2^2)}$$

$$N = \frac{1}{R_1 R_4 (A B - C_2^2)}$$

$$\frac{1}{R_4} = \frac{1}{R_3} + \frac{1}{r_p}$$

Approximate solution:

$$v_1(t) = q\left[\frac{1}{g_m R_4 C_2} e^{s_1 t} + \frac{C_3}{A B} e^{s_2 t}\right]$$

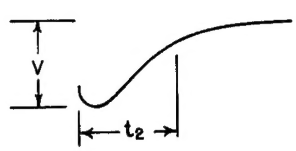
$$v_2(t) = \frac{q}{C_2}\left[- e^{s_1 t} + e^{s_2 t}\right]$$

$$s_1 = -\frac{g_m C_2}{A B}$$

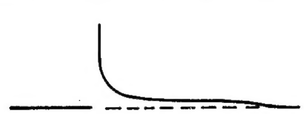
$$s_2 = -\frac{1}{g_m R_4 R_1 C_2}$$

In the derivation the vacuum tube was treated as linear, grid current was neglected, the action of the counter was assumed to be instantaneous, and the leakage resistance across the counter was assumed to be infinite.

If an oscilloscope is connected to the plate of the vacuum tube, pulses of the following shape appear:



Rough measurements have shown that $v = \frac{q}{C_2}$ and that the relaxation time $t_2 = g_m R_4 R_1 C_2$. The shape of the pulses seen with the oscilloscope on the grid of the vacuum tube is as follows:



The predicted constants for this pulse also agree very roughly with measured ones. (Report CP-3609 describes the experiments more completely). For times after ten microseconds the equations derived agree, both with the general features of the pulses observed and, at least within a factor of two, with measured values of amplitudes and relaxation times.

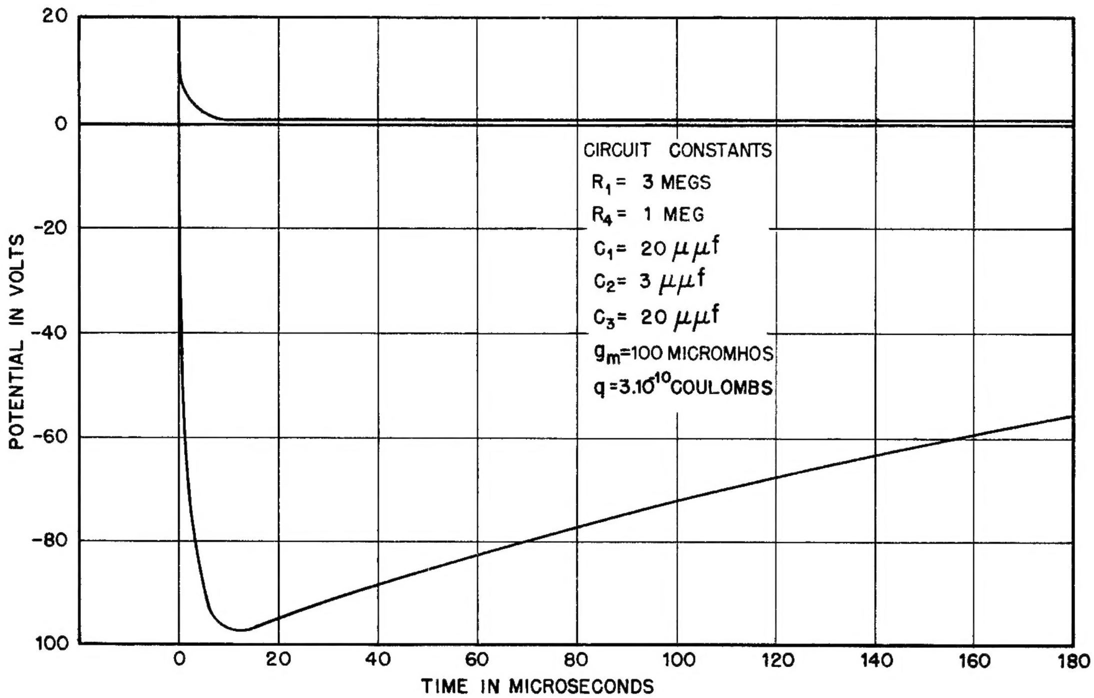


Figure 1. Predicted pulses in a Neher-Harper circuit.

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